JOURNEY TOWARDS PHYSICS AXIOMATIZATION

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ABSTRACT. After more than a century, Hilbert's sixth problem of physics axiomatization is still unsolved. Recent attempts of producing a comprehensive "Theory of Everything", like string theory, has very little chance of obtaining experimental confirmation, and Gödel's incompleteness theorem seems to prohibit a single cohesive axiomatic system, while at the same time nature appears unique and unified. New approaches may be needed to help advance towards a solution of Hilbert's sixth problem. In this paper several axiomatization methods are compared and a systematic research program for solving Hilbert's sixth problem is introduced. A methodology for identifying physical principles is presented as well.

KEYWORDS AND PHRASES: Physics axiomatization, Theory of Everything (ToE), Hilbert's sixth problem, quantum mechanics, special and general relativity, Bourbaki approach to mathematics, axioms and requirements (postulate), composability principle, tensor (Riemann, Minkowski).

I. WHY AXIOMATIZE PHYSICS?

Is axiomatization a useless "goldplate the carburettor" activity which it will not make the car run any better as Irving Segal put it? Why should we not wait first for a final "Theory of Everything" (**ToE**) to be certain we have the correct theory? Gödel incompleteness theorem[2] showed that there is no hope axiomatizing mathematics. Why would physics be any different?

Those are all good questions with hopefully equally good answers. Axiomatizing physics is not a useless exercise because we need to answer *why* some mathematical structures are distinguished

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[2] K. Gödel, "Üeber formal unentscheidbare Sätxe der Principia Mathematica und verwandter Systeme I", *Monatshefte für Mathematik und Physik* **12**, **XXXVIII,** 1931.

by nature like: $SO(3,1)$, $U(1)$, $SU(2)$, $SU(3)$. We also need to know *how* those structures can be combined in a coherent structure. We can certainly wait for the ultimate **ToE**, but maybe we can help get us there faster, and some encouraging results were already obtained. Indeed, mathematics cannot be axiomatized, but nature appears to be unique and unified and if the standard axiomatization approach is forbidden by the incompleteness theorem, then alternative methods need to be investigated.

II. APPROACHES TO AXIOMATIZATION

Several proven approaches to axiomatization are already available. First and foremost, there is the standard approach of defining axioms and deriving mathematical theorems. But this is only one of the choices available. In physics, some theories are introduced as a consequence of a principle of nature and this looks to be similar with the standard approach of picking up postulates and deriving consequences. But it will be shown later that this kind of approach is fundamentally different. Then, there is the Bourbaki approach of starting simple and building up complexity. Other approaches like category theory are available as well but they will not be discussed.

A. Standard axiomatization approach

The standard approach started with Euclidean geometry: define axioms and then prove theorems. The value of the resulting system is given by the value of the axioms and coming up with the right axioms is a very hard problem. It is not every day that a new axiom is proposed, and the process is not constructive. In fact, it requires a high degree of creativity and "cleverness". There is nothing wrong with being smart, but this does not make a workable systematic research program. Typical motivations for axioms are examples from nature, moment of inspiration, agreement with experiments. For example the old quantum mechanics was axiomatized early on by von Neumann and some of his axioms were later found incorrect for the infinite dimensional case.

There is another aspect of the standard approach which is usually overlooked. Picking axioms resembles fencing an area. The axioms of the standard method have an intrinsic boundary direction: they point inward towards the consequences, and the mathematical theorems form a "white box", a "what you see is what you get" structure. However, this is not the only way of building a theory. Let us analyze how designers and engineers build things. Or let us look at how a person does a mundane task: buying a car. When you buy a car you do not start with metallurgical and chemical axioms, you have requirements: colours ***x***, ***y***, and ***z*** are acceptable, price to be in this range, the car has to have this many seats, etc. Then from the available choices you filter out what does not correspond to your needs. Similarly, in engineering, one starts with requirements as well and then picks from the available structures and methodologies the best design for the job. But what does this have to do with axiomatization? Axiomatization includes the process of defining a boundary between the theory and the outside world of mathematics. And this boundary could be pointing inward, or outward. If axioms are pointing inward, requirements are nothing but an outward looking boundary which do not derive internal mathematical consequences but reject what does not fit from an existing pool of structures.

B. Principle axiomatization approach

We can now introduce a new way of physics axiomatization, the axiomatization using physical principles. The best examples of this approach are special and general relativity. In special relativity Einstein starts with two postulates, or "requirements of nature": principle of relativity, and the constant speed of light. The theory is then constructed by removing what does not satisfy those requirements and keeping what does. For example the old fashion Galilean transformation is rejected by the constant speed of light requirement. Similarly in general relativity one starts with the equivalence principle between inertial and gravitational mass.

Axiomatizing by physical principles is an outward looking process which rejects the overwhelming number of mathematical structures, and accepts a few structures compatible with the physical principle. This process results in a "black box" where the selected structures are not necessarily unified or have any cohesiveness. In practice one augments the principles with mild technical axioms to arrive at the desired outcome. For example in the special relativity case one additional technical axiom is the linearity of the coordinate transformations.

To be able to successfully axiomatized physics in this approach one needs to have strong physical intuition, but the process is much simpler than the standard approach. Being easier, it is also not complete. In general this approach answers only ***why*** some mathematical structures are selected by nature, not ***how*** can those selected mathematical structure fit each other coherently. So ***why*** is Lorentz transformation a distinguished mathematical structure selected by nature? Because the speed of light is constant. Why is the speed of light constant? This is a physics requirement and a methodology for selecting core physics requirements will be presented later.

Now not everything seems to be able to be put in this axiomatization form, and people tried for long time to axiomatize quantum mechanics in this fashion, but the very definition of what quantum mechanics is seems very elusive. An approach to axiomatizing quantum mechanics in this fashion is presented in the appendix.

Suppose now we have all the nature requirements in front of us and they demand the existence of quantum mechanics and gravity. Constructing a coherent theory of quantum gravity is a very hard project and answering ***why*** it is clearly not enough. We need an approach for a coherent unification of disjoined mathematical structures, in other words we need to answer ***how***.

Fortunately there is such an approach already available for us, it is the Bourbaki approach.

C. The Bourbaki axiomatization approach

Nicolas Bourbaki is a collective name of a group of French mathematicians who proceeded to systematically build mathematics starting from set theory and adding complexity gradually. This helped usher in the modern view of mathematics: mathematics is only about relationships, devoid of any ontological meanings. Take for example the case of imaginary numbers. For more than three hundred years people had serious difficulty to accept what we now take for granted. Even the very name "imaginary" is a relic of the history behind it. How can you take the square root of a negative numbers? Do imaginary numbers really exist? They are defined by something impossible so why should we take them seriously? What finally made the case for complex and imaginary numbers was their two by two matrix representation which has no objectionable ontological status:

$$z = a + ib = a + \sqrt{-1}\,b = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{1}$$

The lesson learned from this simple example is that there are several realizations of the same mathematical structure and makes no sense to attach any ontological meaning to any of them. This suggests then the following axiomatization process: extract all essential characteristics of a domain and then do a systematic search for the actual realization of the ontological-neutral relationships contained in those essential characteristics. This process is free of guesswork, and straightforward making it ideal to tackle hard physics axiomatization problems. Using this process, Emile Grgin pioneered the beginning of the axiomatization process for quantum mechanics as a principle approach followed by a systematic search for concrete realizations[3].

III. COMPARING THE AXIOMATIZATION APPROACHES

We now have all the required puzzle pieces to axiomatization approaches and can proceed towards introducing a systematic research program towards physic axiomatization. First we need to distinguish between *axioms* and *requirements*. They form the boundary between one mathematical area and the rest of the infinite world of mathematics. But they are inward looking deriving mathematical consequences, or outward looking eliminating most of the infinite world of mathematics. In physics, requirements are usually named postulates and they are justified by experimental results. For the same theory, axioms and requirements are usually not the same. For example special theory of relativity can be constructed from the *requirements* of relativity and constant speed of light, or from the *axiom* of the Minkowski metric tensor of the usual (3,1) – signature.

Obtaining the axioms of nature is an extremely hard project because it is hard to guess right and subsequent development can find exceptions. On the other hand obtaining the requirements is much simpler because it requires only physical intuition and agreement with experiments. There are effective theories like quantum field theory which has only a limited range of applicability due to the existence of the so-called Landau pole, but once the range of validity is specified, the requirements do not change with the natural development of the theory.

As an inward looking boundary, axioms lead mathematicians to desire generalizations, while as an outward looking boundary, *requirements* make physicists to seek uniqueness. For the mathematician, generalizations are "an escape from Egypt", but for a physicist generalizations are a "banishment Paradise".

Gödel's incompleteness theorem may prevent the existence of an ultimate ToE in closed standard axiomatic form, but this result has no relevance for the much more lax requirements/principle approach where the final selected mathematical structures may not be

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3 E. Grgin, *The Algebra of Quantions*, Authorhouse, Indiana, 2005.

unified. The requirement approach only answer ***why*** some mathematical structures are used by nature, but we also need to answer ***how*** are the selected structure coexist harmoniously.

To solve the ***how*** problem, any additional axiomatization approach will do, but the Bourbaki approach was selected because it was successfully employed into the beginning of solving the axiomatization of quantum mechanics as a principle approach, a long sought after physics goal. In this approach one first boils down the essential characteristics to bare minimum, and then proceeds at constructing a systematic search for concrete realizations. The first part is achieved through the *requirements*/principle approach, while the second part is a straightforward systematic search free of guesswork.

IV. SEARCHING FOR THE REQUIERMENTS OF NATURE

There is one more key element for the research project to be viable. We need physics intuition to start the **why** phase. It is hard to come up with new physics principle such as the constant of the speed of light because experimentalist already accumulated a large body of knowledge and no stone was left unturned within the current experimental abilities. But we can set our sights much higher and we can play a Lego exercise: "Let's build a universe". In other words, we can pretend to be God for a second. First, what building blocks do I have at my disposal? This is easy: the timeless mathematical structures of the Platonic world of mathematics. And indeed, Wigner famously said: "*the unreasonable effectiveness of mathematics*"[4]. Barring supernatural explanations, reality is only made up of mathematical relationships. So it must be that reality is nothing but math organized differently, not unlike liquid water and ice are made up of the same chemical molecule.

If the Platonic world of mathematics and Nature are basically the same thing organized differently, the natural thing to do is to compare them: "*Identify all mathematical properties of the physical world that are universally valid in the real world and are not universally valid in the abstract world of mathematics*"[5].

So far three principles satisfy this criterion. Two of them were originally discovered as part of unrelated research programs, but

4 E. P. Wigner, "The unreasonable effectiveness of mathematics in the natural sciences. Richard Courant lecture in mathematical sciences delivered at New York University, May 11, 1959", *Communications in Pure and Applied Mathematics*, vol.13, 1, 1960.

5 F. Moldoveanu, *Heuristic rule for constructing physics axiomatization,* arXiv: 1001.4586v1

it makes sense to introduce them together from the requirements point of view.

In the abstract world of mathematics, truth is a property confined to an axiomatic system. Moreover, an axiomatic system cannot define its truth property itself. But as God I want to create an ontology, a reality whose truth value is freed from the axiomatic boundary and it is universal. It is this universality which allowed the Galilean revolution to occur and put experiments at the centre of settling questions about nature. Hence the first principle is: universal truth property, or universal non-contextuality of the truth value of facts of nature[6]. Because of this, we usually have two meanings of truth: true as a logical consequence, or true as something corresponding to reality.

Looking again at the abstract world of mathematics, and specifically at Gödel's incompleteness theorem we see that in here things are not unified, but separated and "frozen". This is not what we want. The *requirement* would be to have a cohesive nature with no "island universe" following different natural laws. The second principle is the "*composability principle*": any two physical systems obeying physical laws obey the same law when combined into the same system[7]. This principle is actually extremely powerful as very few mathematical structures can obey it. In fact there are only three: classical mechanics, quantum mechanics, and all Lie groups.

In any axiomatic system, the algorithmic information content is finite. The last requirement for nature is to obey a principle of infinite complexity. In its original form this principle was called the "deformability" principle[8].

In summary, the three principles/*requirements* of nature coming from comparing nature with the abstract world of mathematics are: **universal truth property**, **composability**, **infinite complexity**. The rest of solving Hilbert's sixth problem is deriving mathematical consequences out of them in a systematic way.

This research program is already in progress with many encouraging results, but it is only in its early infancy. For example, with the help of some technical axioms, it can be shown that the first principle

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6 *Ibidem*.

7 E. Grgin and A. Petersen**,** "Algebraic Implications of Composability of Physical System", *Communications in Pure and Applied Mathematics*, vol. 50, nr. 177, 1976.

8 J. Rau, "On the Metric Structure of Space-Time", in M.A. del Olmo, M. Santander, and J. Mateos Guilarte eds., *Group Theoretical Methods in Physics,* Vol. II, Proc. XIX Int. Colloquium, Salamanca, Spain, June 29-July 4, 1992 (Anales de Fisica, Monografias 1, CIEMAT, Madrid, 1993), pp. 483–486.

demands the existence of causality and time, the second principle demands quantum mechanics, and the third principle demands the existence of a Riemann metric tensor. Combining the first and third principle demands special theory of relativity with an unspecified number of spatial dimensions. Combining special relativity and quantum mechanics can only be consistently done in a 3+1 spacetime as shown by the systematic search for realizations for quantum mechanics in the algebraic formalism. Electroweak gauge symmetry can also be obtained as a necessity to unify relativity and quantum mechanics. The mid-term goal is to eliminate as much as possible the additional technical axioms and strengthen the results obtained so far.

The major challenge on this framework is the explanation for the beginning of time, the existence of a multiverse, and quantum gravity. The fact that nature is unique as a consequence of physics principles is a double edge sword. As Guth put it, why is this universe happening only once? This seems to be at great odds with the Copernican ideas that we are in no way special.

We may need to look beyond the three principles/*requirements* and it looks that the best approach to solve the issues above it is using the Darwinian principle of the survival of the fittest: proving there is a vacuum and demanding its stability from a sea of all potential multiverses which do not respect universal truth property or maybe composability. At this time those are only speculations, and there are a lot of tractable standard problems to be solved in the outlined research project.

Solving Hilbert's sixth problem will not change the Galilean nature of physics and experiments will remain the way of testing agreement with nature. This is because the three principles have to pass all past, present, and future experimental tests. We only pretended to play God, we cannot write the equations of nature on paper, say "fly" and a new universe will be born. We can construct virtual realities on computers but they do not satisfy the third principle, infinite complexity. Alternative ontologies like computer games, virtual reality, even cartoon characters have no less right to exist than our universe, there is no ontological hierarchy and the lack of a criterion to rank ontologies means that we have we have an ontological democracy. The question "*why there is something rather than nothing*" has a simple answer: because it can be. In any ontological universe there is a simple test for the existence of a creator: is the information conserved or not? If no, the "laws of nature" in that universe, are not complete and require an outside entity. In other words, God can exist only as "God of the gaps".

V. CONCLUSION

A research project aimed at solving Hilbert sixth problem is introduced. This research projects distinguishes between inward looking axioms and outward looking requirements/principles. It aims at explaining both why some mathematical structures play a key role in nature, and how they can be combined in a coherent structure. Three principles of nature: universal truth property, composability, and infinite complexity were presented. Those principles were originally introduced by different motivations and they already generated mathematical consequences. The problems of beginning of time, uniqueness of our universe, or quantum gravity are not part of this framework and new ideas needs to be introduced like Darwinian survival of the fittest and Copernican ideas of not being special. However, unlike the prior three principles, no mathematical consequences were yet derived out of those principles, and their usefulness is only speculative at this point. (There are very good reasons to select the additional Copernican and Darwinian principles. For example Darwinian survival of the fittest can explain the emergence of classical reality from quantum mechanics[9]. The Copernican principle seems to imply the existence of a multiverse which is compatible with string and inflation theory.)

The strongest principle so far is the composability principle which can be used to define quantum mechanics as shown in the appendix. *The problem is not completely solved*, as this principle allows additional mathematical structures and the search for additional criteria to isolate only quantum mechanics continues.

VI. APPENDIX: CONSEQUENCES OF THE COMPOSABILITY PRINCIPLE

In this section a high level introduction into using the composability principle to define quantum mechanics is given. Quantum mechanics can be introduced in many formalisms, but the preferred approach is that of Hamiltonian mechanics and of the C^* algebraic approach. This summary follows the core results of Emile Grgin[10] who introduced the concept of a two-algebra approach to quantum mechanics.

Historically, the idea was to search for a common axiomatization of both classical and quantum mechanics because whatever they have

9 W. H. Zurek, *Quantum Darwinism*, arXiv: 0903.5082v1

10 E. Grgin and A. Petersen, *ibidem*.

in common must be absolutely essential. Both classical and quantum mechanics have two products, one symmetric and one anti-symmetric. For classical mechanics the two products are the regular function multiplication and the Poisson bracket. For quantum mechanics, the products are the Jordan and the Lie products. There is a one to one correspondence between them, usually called the dynamic correspondence between observables and generators. Mathematically this corresponds to a multiplication by $\sqrt{-1}$ mapping hermitean into anti-hermitean operators, and physically this corresponds to the uncertainty principle.

Let us call S_1 and A_1 the symmetric and the anti-symmetric products of system one, S_2 and S_T the correspondent products of system two, and S_T and A_T the products of the composed system. Composability demands the following:

$$S_T = S_1 \otimes S_2 - a \cdot A_1 \otimes A_2 \quad (2)$$

$$A_T = S_1 \otimes A_2 + A_1 \otimes S_2 \quad (3)$$

with $a = +1$, 0, −1. Then $a = +1$ corresponds to quantum mechanics ($a = \hbar$), and $a = 0$ corresponds to classical mechanics, and $a = -1$ corresponds to either a split-complex quantum mechanics or all Lie groups. The symmetric and anti-symmetric products also obey three identities: Lie, Leibniz, and associated identity mapping commutators to anti commutators. This formalism is nothing but the C^* algebraic formalism without the norm positivity condition. From here one can determine their concrete realizations resulting in the usual unitary groups, and also in some additional exceptional cases like $SO(2, 4)$ corresponding to the conformal compactification of $SO(1, 3)$ leading to the Dirac's equation and the electroweak symmetry $U(1) \times SU(2)$[11]. Those are advanced topics beyond the scope of this paper.

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